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Optimal Default and Liquidation with Tangible Assets and Debt Renegotiation*

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1 Introduction

The purpose of this paper is to propose a new pricing model for corporate securities issued by a levered firm with the possibility of debt renegotiation. We take the structural approach that the firm's earnings follow a geometric Brownian motion (GBM for short) with stochastic collaterals. As in Leland (1994), equity holders can default the firm for their own benefits, when the earnings become insufficient to go on the firm. In addition, equity holders may want to liquidate the firm by repaying the face value of debt to debt holders in order to get enough residuals, when the value of collaterals becomes sufficiently high. Unlike the existing structural models, the bivariate structure of our model can not only capture realistic credit spreads observed in the market, but also explain many empirical findings reported in the literature.

An important development in the structural approach was made in a seminal paper by Leland (1994), who considers the optimal capital structure of a firm based on the balancing theory, i.e., the trade-off between default costs and tax benefits. Assuming that the firm's asset value follows a GBM and corporate securities are contingent claims written on the asset value, Leland (1994) derived the optimal default boundary and the equity value simultaneously by solving a free-boundary problem, because the default boundary is chosen by equity holders so as to maximize the equity value. The debt and total firm values are calculated accordingly. The optimal coupon rate (and hence credit spread) is then determined so as to maximize the total firm value.

However, as Mella-Barral (1999) pointed out, credit spreads calculated by the Leland model are close to those observed in the market only for significantly high default costs. Also, Eom et al. (2004) noted that the models by Merton (1974) and Geske (1977) underestimate the credit spreads, while the models by Longstaff and Schwartz (1995), Leland and Toft (1996) and Collin-Dufresne and Goldstein (2001) overestimate. Since then, several attempts have been made to overcome the deficiency in the structural approach. Among them, Anderson and Sundaresan

*This paper is an abbreviated version of Goto et al. (2010).

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(1996) and Mella-Barral and Perraudin (1997) proposed a structural model with debt renegotiation. In reality, debt is renegotiated, because liquidation is costly and debt holders cannot suffer from liquidation. Hence, equity holders have an incentive to renegotiate in order to reduce the contractual debt service.

Following Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997), Mella-Barral (1999) considered a model with departures from absolute priority rule, while Fan and Sundaresan (2000) incorporated the medium bargaining power and provided the Nash bargaining solution. Structural pricing models with debt renegotiation suggest that, when creditors have little bargaining power, a large part of credit spreads may be due to the possibility of strategic default risk. Debt renegotiation by strategic debt service provides higher credit spreads, whereby the models mentioned above succeed to generate realistic credit spreads.

However, recent empirical studies such as Davydenko and Strebulaev (2007) pointed out that the contribution of strategic debt service to credit spreads suggested by the theoretical models is too large. In fact, Davydenko and Strebulaev (2007) found that bond prices do appear to be affected by the possibility of debt renegotiation, while their quantitative contribution to both the average level and the cross-sectional level of credit spreads is below transactions costs.[1] Based on this result, they are inclined to suggest that debt holders are likely to have significant bargaining power, which limits equity holders' strategic behavior. Also, Acharya et al. (2006) introduced the additional option that firms can carry cash reserves as protection against costly liquidation and concluded that debt renegotiation typically has a negligible effect on the yield spreads under the model of Anderson and Sundaresan (1996).

There is another defect in the theoretical models mentioned above. As Hart and Moore (1994, 1998) emphasize, it is important to distinguish strategic default from liquidity default. While *liquidity default* occurs when the firm's cash flows are insufficient to cover the debt service, *strategic default* occurs when the firm fails to pay full amount of debt service in debt contract even though it possesses the resource to do so. Most of the papers in the debt renegotiation literature distinguish strategic default from liquidation, but neglect liquidity default.[2] Moreover, in those models, model parameters determine which occurs, either liquidation or strategic default, in advance. In other words, we know *a priori* whether or not debt renegotiation eventually occur by looking at the parameter values, and there is no uncertainty how the firm terminates in the future.

In this paper, we extend the existing models to the bivariate framework by introducing the value of tangible assets, which plays the role of collaterals. Because of the bivariate feature, we can distinguish strategic default, liquidity default and the ordinary liquidation. It is shown that, in our model, liquidity default can occur when the value of collaterals is relatively high but the value of earnings (more precisely, earnings before interest and tax; EBIT for short) is substantially low, irrespective of the bargaining power of equity holders. In this case, the firm

[1] Davydenko and Strebulaev (2007) also found the fact that the bond prices are likely to be affected by the possibility of debt renegotiation, especially when the costs of liquidation are likely to be high and credit quality of the issuer is relatively low.

[2] Fan and Sundaresan (2000) consider liquidity default; however, it means liquidation by impossibility of coupon payment, which is basically the same as liquidation considered in other papers.

must pay large maintenance costs for tangible assets despite of the poor business performance, and equity holders cannot afford to carry out the debt contract, resulting in liquidity default. When the value of tangible assets is sufficiently high and the EBIT is substantially low, our model selects liquidation, because equity holders want to repay the face value of debt to debt holders in order to get enough residuals. Otherwise, the firm terminates as strategic default; however, the pattern depends on the sample path of the bivariate process, not the initial parameter values. This means that equity holders can select either liquidity default or liquidation, depending on the economical condition, without entering debt renegotiation. This is a significant difference between our model and the other existing models in the debt renegotiation literature.

The possibility of liquidity default and liquidation without entering debt renegotiation is quite important from the pricing perspectives. If renegotiation always occurs in a given model, the effect of strategic debt service will be overstated and its contribution to credit spreads evaluated from the model becomes too high, which can explain the empirical findings in Davydenko and Strebulaev (2007). In fact, our model can produce credit spreads consistent with the empirical findings even when equity holders have full bargaining power, as expected. The contribution of strategic debt service to default premium depends on the underlying variables, in contrast to the existing models such as Mella-Barral and Perraudin (1997).

2 Debt and Agency Costs

We consider a firm with a set of tangible (or physical) assets that can yield revenues. Both the value of tangible assets, V, and the firm's EBIT (earnings before interest and tax), P, are modeled with correlation in a dynamic setting.[3] The instantaneous risk-free interest rate is assumed to be constant and denoted by r. Since we focus on the change of default strategy by introducing stochastic collaterals rather than the capital structure, we neglect the tax benefit and assume that corporate tax rate is zero for simplicity.

2.1 Basic Assumptions

Suppose that the asset value V follows a geometric Brownian motion (GBM):

$$\frac{\mathrm{d}V(t)}{V(t)} = \mu_v \mathrm{d}t + \sigma_v \mathrm{d}B_1(t), \qquad V(0) = v,$$

where μ_v is the instantaneous expected growth rate of V, σ_v is the associated volatility, and B_1 is a standard Brownian motion. The asset needs to be maintained by expending a proportional cost ηV.[4]

On the other hand, the firm's EBIT P is assumed to follow another GBM:

$$\frac{\mathrm{d}P(t)}{P(t)} = \mu_p \mathrm{d}t + \sigma_p \mathrm{d}B_2(t), \qquad P(0) = p,$$

[3] Many papers such as Mella-Barral and Perraudin (1997) and Mella-Barral (1999) consider a single state variable that is supposed to be positively correlated to revenues. In this paper, we explicitly consider both tangible assets and revenues with correlation.

[4] As we explain later, equity holders have an incentive to supply short-term funds by increase in capital even if the EBIT is substantially low.

where μ_p is the instantaneous expected growth rate of P, σ_p is the volatility, and B_2 is another standard Brownian motion with instantaneous correlation $\mathbb{E}[\mathrm{d}B_1\mathrm{d}B_2] = \rho\mathrm{d}t$.[5]

Note that our model can be seen as a bivariate extension of existing models in the literature. For example, if we replace the asset value V and the firm's EBIT P by a constant scrap value γ and earnings $p - w$, respectively, and neglect the maintenance cost ηV of tangible assets, then our model is reduced to the one considered in Mella-Barral and Perraudin (1997).

Throughout the paper, we assume that μ_v, $\mu_p < r$ in order to ensure the existence of value functions of interest. Furthermore, for a levered firm, we assume the following.

Assumption 1 (Levered Firm) *The firm issues a perpetual debt with contractual coupon rate c, face value c/r and collateral $\underline{C}(t) = \min\{V(t), c/r\}$. Moreover,*

1. *Equity holders can voluntarily default the firm. Upon default, debt holders own the residual assets and take over the firm as existing equity holders.*
2. *Equity holders can liquidate the firm's tangible assets and repay the collateral to debt holders. In this case, the firm cannot go on.*
3. *The firm cannot redeem the debt. That is, the firm cannot turn back to a pure equity firm unless it experiences a default.*

Hence, equity holders have options either to default or to liquidate the firm, depending on the state of the variables (V, P). If the firm is defaulted, debt holders take over the firm and it becomes a pure equity firm. If the firm is liquidated, debt holders receive the collateral $\underline{C}$ and equity holders will get the residual. Note that, after the firm becomes a pure equity firm, debt holders can liquidate the firm either immediately or after some time, depending on the state of the variables (V, P). We shall explain how default and liquidation occur in this setting later.

We assume that the liquidation of the firm's tangible asset is induced by M&A. As pointed out by Lambrecht and Myers (2007), liquidation is often accompanied by (and apparently forced by) M&A in declining industries. Let $Y(t)$ be the buyout price at time t, and suppose that the M&A market is perfectly competitive and equity holders have full bargaining power against all the raiders. Then, under these assumptions, the raiders' return is given by $V(t) - \underline{C}(t) - Y(t) = 0$ and the M&A occurs at the best time for equity holders.[6] In other words, the liquidation can occur at the value

$$Y(\tau) = V(\tau) - \underline{C}(\tau) \tag{1}$$

for any stopping time τ.

Because this assumption is essential for the valuation of corporate securities, we state it formally as the following assumption.

[5]Throughout this paper, we fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote the expectation operator by $\mathbb{E}$. The canonical filtration generated by the underlying stochastic structure is denoted by $\{\mathcal{F}_t\}$, where $\mathcal{F}_t$ defines the information available at time t.

[6]It is an interesting problem to discuss the M&A under the assumption that raiders choose the acquisition timing so as to maximize their return under imperfect competition. We leave this interesting problem as a future work.

Assumption 2 (Liquidation and M&A) *The liquidation of the firm's tangible asset is induced by M&A, where the M&A market is perfectly competitive and equity holders have full bargaining power against all the raiders. The value of the firm's tangible asset upon liquidation is given by (1) for any stopping time τ.*

2.2 Pure Equity Firm

Before preceding, we first consider a pure equity firm as a benchmark to the levered firm. To do so, we denote by W^* the equity value of the firm without debts (hence, W^* is equal to the firm value). Note that equity holders can receive the EBIT P minus the maintenance cost ηV as dividends. Hence, they will liquidate the firm against either a decrease in profits or defrayment of maintenance cost, and upon liquidation, they receive V as the liquidation payoff.

Suppose that $P(0) = p$ and $V(0) = v$, and consider the value function $W^*(p, v)$ of the pure equity firm. Let τ_0 be the liquidation time chosen by equity holders to maximize their own value. The value function is given by

$$W^*(p,v) = \sup_{\tau_0 \in \mathcal{T}_0} \mathbb{E}\left[\int_0^{\tau_0} \mathrm{e}^{-rt}(P(t) - \eta V(t))\mathrm{d}t + \mathrm{e}^{-r\tau_0} V(\tau_0)\right], \tag{2}$$

where $\mathcal{T}_0$ denotes the set of admissible stopping times in $[0, \infty)$.

In order to preclude arbitrage opportunities, it is well known that the following equilibrium condition must hold:

$$rW^* = p - \eta v + \mathbb{E}\left[\frac{\mathrm{d}W^*}{\mathrm{d}t}\right].$$

Hence, applying Ito's formula, we obtain the partial differential equation (PDE for short)

$$\mathcal{A}W^*(p,v) + p - \eta v = 0, \tag{3}$$

where the partial differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}W^*(p,v) = \frac{1}{2}p^2\sigma_p^2 W_{pp}^* + \frac{1}{2}v^2\sigma_v^2 W_{vv}^* + pv\sigma_p\sigma_v\rho W_{pv}^* + \mu_p p W_p^* + \mu_v v W_v^* - rW^*. \tag{4}$$

Note that the PDE (3) has no constant term and the payoff can be represented in terms of p/v only. Hence, we can find the value function $W^*(p, v)$ analytically by using the change-of-variable $z = p/v$.

Proposition 1 (Pure Equity Firm) *The value function of the pure equity firm is given by*

$$W^*(p,v) = \begin{cases} \dfrac{p}{r-\mu_p} - \dfrac{\eta v}{r-\mu_v} + \dfrac{(r-\mu_v+\eta)v}{(1-\lambda)(r-\mu_v)}\left(\dfrac{p}{b^* v}\right)^{\lambda}, & \text{for } \dfrac{p}{v} > b^*, \\ v, & \text{for } \dfrac{p}{v} \le b^*, \end{cases} \tag{5}$$

where λ is the negative root of the quadratic equation

$$\frac{1}{2}\sigma_z^2\lambda(\lambda-1) + (\mu_p - \mu_v)\lambda - (r - \mu_v) = 0$$

with $\sigma_z^2 = \sigma_p^2 + \sigma_v^2 - 2\rho\sigma_p\sigma_v$. *The optimal liquidation time is obtained as*

$$\tau_0^* = \inf\{t \geq 0 : P(t)/V(t) \leq b^*\},$$

where the liquidation threshold is given by

$$b^* = \frac{\lambda}{\lambda - 1}\frac{r - \mu_p}{r - \mu_v}(r - \mu_v + \eta). \tag{6}$$

The variable $z = p/v$ can be interpreted as the return on tangible assets (ROTA). According to Proposition 1, the pure equity firm will continue its operation as long as ROTA stays above the threshold b^*. When ROTA goes down to the threshold b^*, the pure equity firm is liquidated and equity holders receive the value v of tangible assets.

2.3 Equity Value of the Levered Firm

This section derives the equity value of the levered firm under Assumption 1. In our setting, equity holders can select one of the two stopping times, default time τ_1 or liquidation time τ_2, so as to maximize their equity value. Recall from the second statement in Assumption 1 that equity holders can repay the collateral to debt holders and receive the residual upon liquidation. On the other hand, they will receive nothing if the firm is defaulted.

In order to make the difference between the levered firm and the pure equity firm explicit, we denote the value functions of the levered firm by using a hat.[7] For example, the equity value of the levered firm is denoted by $\hat{F}(p, v)$. Under Assumption 1, the equity value is given by

$$\hat{F}(p, v) = \sup_{\tau_1, \tau_2 \in \mathcal{T}} \mathbb{E}\left[\int_0^{\tau_1 \wedge \tau_2} e^{-rt}(P(t) - \eta V(t) - c)dt \right.$$
$$\left. + \mathbf{1}_{\{\tau_1 > \tau_2\}} e^{-r\tau_2} \max\{V(\tau_2) - \underline{C}(\tau_2), 0\}\right], \tag{7}$$

where c denotes the coupon payment and $\mathcal{T}$ the set of admissible stopping times in $[0, \infty) \times [0, \infty)$. Note that the optimal stopping times are chosen by looking at the state variables (P, V). Also, since liquidation is selected only when the payoff is greater than that of default, we obtain

$$\max\{V(\tau_2) - \underline{C}(\tau_2), 0\} = V(\tau_2) - \frac{c}{r} > 0. \tag{8}$$

Note that, if $V(\tau_2) - c/r = 0$, this means default.

Because the underlying process (P, V) is homogeneous both in time and in space and the value function is given by (7), we can divide the whole state space $(0, \infty) \times (0, \infty)$ of (P, V) into the three mutually exclusive regions: ordinary operation $\hat{\mathcal{C}}$, default $\hat{\mathcal{D}}$ and liquidation $\hat{\mathcal{L}}$. Associated with each region is the PDE satisfied by the value function $\hat{F}(p, v)$. That is,

$$\begin{cases} \hat{F}(p, v) = v - \dfrac{c}{r}, & \text{for } (p, v) \in \hat{\mathcal{L}}, \\ \mathcal{A}\hat{F}(p, v) + p - \eta v - c = 0, & \text{for } (p, v) \in \hat{\mathcal{C}}, \\ \hat{F}(p, v) = 0, & \text{for } (p, v) \in \hat{\mathcal{D}}, \end{cases} \tag{9}$$

[7]In the next section, we shall denote the value functions of a levered firm with the possibility of debt renegotiation with no accent marks.

where $\mathcal{A}$ is the operator defined by (4).

The three regions in Equation (9) are not yet determined. In other words, we must determine the boundaries between $\hat{\mathcal{C}}$ and $\hat{\mathcal{L}}$ and between $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ simultaneously when we solve the PDE (9). Hence, our problem is the so-called *free-boundary problem* in the two-dimensional setting.

It is well known (see, e.g., Fleming and Soner, 1993) that a function satisfying the PDE (9) as well as the value-matching and smooth-pasting conditions is the value function $\hat{F}(p,v)$. More specifically, denote the boundary between $\hat{\mathcal{C}}$ and $\hat{\mathcal{L}}$ by $\hat{\mathcal{B}}_{CL}$. Then, the value-matching condition is given by

$$\hat{F}(p,v) = v - \frac{c}{r}, \qquad (p,v) \in \hat{\mathcal{B}}_{CL}, \tag{10}$$

and the smooth-pasting condition is stated as

$$\hat{F}_p(p,v) = 0, \qquad \hat{F}_v(p,v) = 1, \qquad (p,v) \in \hat{\mathcal{B}}_{CL}.$$

For the boundary $\hat{\mathcal{B}}_{CD}$ between $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$, the following conditions hold:

$$\hat{F}(p,v) = 0, \quad \hat{F}_p(p,v) = 0, \quad \hat{F}_v(p,v) = 0, \quad (p,v) \in \hat{\mathcal{B}}_{CD}. \tag{11}$$

The PDE (9) must be solved under these conditions.

The difficulty to solve the PDE (9) arises in our setting. Because of the constant coupon rate c, we cannot apply the change-of-variable method, $z = p/v$, which was successfully used in the pure equity case (i.e., $c = 0$). In the following numerical examples, we solve the PDE (9) with value-matching and smooth-pasting conditions by using a finite difference method. To this end, it is helpful to obtain the boundary conditions for $\hat{F}(p,v)$ when p and/or v tend to either 0 or infinity. For these results, see Goto et al. (2010).

2.4 Debt and Firm Values of the Levered Firm

As stated in Assumption 1, debt holders receive coupon payments c until either default epoch τ_1 or liquidation time τ_2, whichever happens first. If equity holders decide to liquidate before defaulting the firm, debt holders receive the face value c/r at τ_2 from Equation (8). On the other hand, if equity holders decide to default the firm, debt holders will take over the firm at τ_1 and the firm becomes a pure equity firm. The debt holders will then receive the profit ξp, where $\xi < 1$ is the efficiency loss of EBIT p, until they liquidate the firm at time τ_3. The liquidation time τ_3 is determined so as to maximize the value of the pure equity firm.

More specifically, let $X(p,v)$ represent the value of the firm taken over by debt holders, when $P(0) = p$ and $V(0) = v$. Because the firm is a pure equity firm, we can find $X(p,v)$ analytically as

$$X(p,v) = \sup_{\tau_3 \in \mathcal{T}_0} \mathbb{E}\left[\int_0^{\tau_3} \mathrm{e}^{-rt}(\xi P(t) - \eta V(t))\mathrm{d}t + \mathrm{e}^{-r\tau_3} V(\tau_3)\right], \tag{12}$$

$$= \begin{cases} \dfrac{\xi p}{r-\mu_p} - \dfrac{\eta v}{r-\mu_v} + \dfrac{(r-\mu_v+\eta)v}{(1-\lambda)(r-\mu_v)}\left(\dfrac{p}{\underline{b}v}\right)^{\lambda}, & \text{for } \dfrac{p}{v} > \underline{b}, \\ v, & \text{for } \dfrac{p}{v} \le \underline{b}, \end{cases} \tag{13}$$

where λ is given as in Proposition 1 and the liquidation threshold is given by

$$\underline{b} = \frac{\lambda}{\lambda - 1} \frac{r - \mu_p}{r - \mu_v} \frac{r - \mu_v + \eta}{\xi}, \tag{14}$$

which should be compared with the threshold given in (6). The debt value is now obtained as

$$\begin{aligned}
\hat{D}(p,v) &= \sup_{\tau_3 \in \mathcal{T}_0} \mathbb{E}\bigg[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt} c\mathrm{d}t + \mathbf{1}_{\{\tau_1 > \tau_2\}} \mathrm{e}^{-r\tau_2} \frac{c}{r} \\
&\qquad + \mathbf{1}_{\{\tau_1 \le \tau_2\}} \left(\int_{\tau_1}^{\tau_3} \mathrm{e}^{-rt} (\xi P(t) - \eta V(t)) \mathrm{d}t + \mathrm{e}^{-r\tau_3} V(\tau_3) \right) \bigg] \\
&= \mathbb{E}\bigg[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt} c\mathrm{d}t + \mathbf{1}_{\{\tau_1 > \tau_2\}} \mathrm{e}^{-r\tau_2} \frac{c}{r} + \mathbf{1}_{\{\tau_1 \le \tau_2\}} \mathrm{e}^{-r\tau_1} X(P(\tau_1), V(\tau_1)) \bigg].
\end{aligned}$$

Of course, the debt value $\hat{D}(p,v)$ satisfies the following PDE:

$$\begin{cases} \hat{D}(p,v) = \dfrac{c}{r}, & \text{for } (p,v) \in \hat{\mathcal{L}}, \\ \mathcal{A}\hat{D}(p,v) + c = 0, & \text{for } (p,v) \in \hat{\mathcal{C}}, \\ \hat{D}(p,v) = X(p,v), & \text{for } (p,v) \in \hat{\mathcal{D}}. \end{cases} \tag{15}$$

Since the boundaries are known, this is not a free-boundary problem and can be solved numerically by, e.g., the ordinary finite difference method.

Finally, the value of the levered firm is obtained as

$$\begin{aligned}
\hat{W}(p,v) &= \hat{F}(p,v) + \hat{D}(p,v) \qquad (16) \\
&= \mathbb{E}\bigg[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt} (P(t) - \eta V(t)) \mathrm{d}t + \mathbf{1}_{\{\tau_1 \le \tau_2\}} \mathrm{e}^{-r\tau_1} X(P(\tau_1), V(\tau_1)) + \mathbf{1}_{\{\tau_1 > \tau_2\}} \mathrm{e}^{-r\tau_2} V(\tau_2) \bigg],
\end{aligned}$$

where the stopping times τ_1 and τ_2 are selected by equity holders and τ_3 by debt holders.

Remark 1 If we assume a tax benefit for coupon payments, we can consider the optimal capital structure of the firm as in Leland (1994), where the optimal coupon level is selected so as to maximize the firm value $\hat{W}(p,v)$. However, since the boundaries $\mathcal{B}_{CL}$ and $\mathcal{B}_{CD}$ depend on the model parameters such as coupon rate c, in order to obtain the optimal capital structure of the firm, we need to solve the free-boundary problem repeatedly until the firm value is maximized with respect to the coupon rate c. This is not an easy numerical problem, although not impossible.

3 Pricing with Renegotiation

In this section, we consider the possibility of debt renegotiation as in Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997). That is, suppose that debt holders and equity holders can renegotiate about coupon payments each other. Throughout this section, we assume that equity holders have full bargaining power, unless stated otherwise. In the word of Mella-Barral and Perraudin (1997), equity holders can make the 'take-it-or-leave-it' offer to debt holders.

Let $s(p,v)$ denote the service flow function, when strategic debt service is performed. This means that equity holders can offer debt holders a discount of coupon payments instead of default. The offered coupon rate in the state (p,v) is represented by the function $s(p,v)$, which is determined by equity holders, because they have full bargaining power. It is assumed that the service flow function is piecewise right-continuous.

Under the possibility of debt renegotiation, the equity and debt values are different from those without renegotiation. We denote by $F(p,v)$ and $D(p,v)$, respectively, the equity and debt values under renegotiation. Recall that $X(p,v)$ represents the equity value after debt holders take over the firm.

As default approaches, the debt value $D(p,v)$ becomes strictly larger than $X(p,v)$. Equity holders exploit the difference and offer a discount of coupon until $D(p,v)$ becomes equal to $X(p,v)$. Because debt holders have no bargaining power, they must accept this offer. Hence, the service flow function $s(p,v)$ is determined such that $D(p,v) = X(p,v)$. See Mella-Barral and Perraudin (1997) for details.

As before, we define the three mutually exclusive regions in (p,v): ordinary operation $\mathcal{C}$, default $\mathcal{D}$, and liquidation $\mathcal{L}$. In addition, we need to define the region in which equity holders carry out the strategic debt service. This region is denoted by $\mathcal{S}$, which is disjoint from the other regions. Hence, under the possibility of debt renegotiation, we have four mutually exclusive regions, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{S}$. When the process (P,V) is in $\mathcal{C}$, equity holders pay the contractual coupon c. Once (P,V) hits the region $\mathcal{S}$, equity holders offer a discount of coupon instead of default. After that, equity holders default when (P,V) reaches the region $\mathcal{D}$. Summarizing, we assume the following for the case of debt renegotiation.

Assumption 3 (Strategic Debt Service) *If equity holders can make the 'take-it-or-leave-it' offer to debt holders regarding debt service, then there exist two mutually exclusive regions $\mathcal{S}$ and $\mathcal{D}$ such that:*

1. *Default occurs when the process (P,V) first hits the region $\mathcal{D}$,*
2. *For all $(p,v) \in \mathcal{S}$, we have $s(p,v) < c$ so that $D(p,v) = X(p,v)$,*
3. *For all $(p,v) \in \mathcal{C}$, we have $s(p,v) = c$.*

We note that there is still a possibility of liquidation even in the presence of debt renegotiation, because equity holders want to liquidate the firm before default if the tangible asset value V is sufficiently high. This is a significant difference in our model from Mella-Barral and Perraudin (1997). Also, from Assumption 3 and Equation (13), the service flow function $s(p,v)$ must be given by

$$s(p,v) = \begin{cases} c, & \text{for } (p,v) \in \mathcal{C}, \\ \xi p - \eta v, & \text{for } (p,v) \in \mathcal{S}_1 := \mathcal{S} \cap \{p/v > \underline{b}\}, \\ (r-\mu_v)v, & \text{for } (p,v) \in \mathcal{S}_2 := \mathcal{S} \cap \{p/v \le \underline{b}\}, \end{cases} \tag{17}$$

in order to ensure the equality $D(p,v) = X(p,v)$, where $\underline{b}$ is given by (14). Note that the renegotiation region $\mathcal{S}$ is divided into two sub-regions $\mathcal{S}_1$ and $\mathcal{S}_2$.

3.1 Debt and Equity Values

As before, we denote the time of default by τ_1 and the time of liquidation by τ_2. In the presence of debt renegotiation, debt holders will receive coupon payments $s(p,v)$ until either default epoch τ_1 or liquidation time τ_2, whichever happens first. If equity holders decide to liquidate the firm before default, debt holders receive the face value c/r at the liquidation time τ_2. If they decide to default, the firm is taken over by debt holders. The strategic debt service is performed while the process (P,V) stays in the region $\mathcal{S}$. Hence, the equity and debt values are given, respectively, by

$$F(p,v) = \sup_{\tau_1,\tau_2 \in \mathcal{T}} \mathbb{E}\left[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt}(P(t) - \eta V(t) - s(P(t),V(t)))\mathrm{d}t + \mathbf{1}_{\{\tau_1 > \tau_2\}}\mathrm{e}^{-r\tau_2}\left(V(\tau_2) - \frac{c}{r}\right)\right],$$

and

$$D(p,v) = \mathbb{E}\left[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt}s(P(t),V(t))\mathrm{d}t + \mathbf{1}_{\{\tau_1 > \tau_2\}}\mathrm{e}^{-r\tau_2}\frac{c}{r} + \mathbf{1}_{\{\tau_1 \le \tau_2\}}\mathrm{e}^{-r\tau_1}X(P(\tau_1),V(\tau_1))\right],$$

where $\mathcal{T}$ denotes the set of admissible stopping times in $[0,\infty) \times [0,\infty)$.

It is well known that the equity value $F(p,v)$ satisfies the PDE

$$\begin{cases} F(p,v) = v - \dfrac{c}{r}, & \text{for } (p,v) \in \mathcal{L}, \\ \mathcal{A}F(p,v) + p - \eta v - c = 0, & \text{for } (p,v) \in \mathcal{C}, \\ \mathcal{A}F(p,v) + (1-\xi)p = 0, & \text{for } (p,v) \in \mathcal{S}_1, \\ \mathcal{A}F(p,v) + p - (\eta + r - \mu_v)v = 0, & \text{for } (p,v) \in \mathcal{S}_2, \\ F(p,v) = 0, & \text{for } (p,v) \in \mathcal{D}, \end{cases} \tag{18}$$

where the partial differential operator $\mathcal{A}$ is defined in (6). Because the boundaries between the regions are not known in advance, the problem to solve the PDE (18) is a free-boundary problem.

Note again that the second equation in (18) has no closed-form solution due to the constant term c. In contrast, the fourth equation has a closed-form solution and the boundary $\mathcal{B}_{\mathcal{DS}_2}$ between $\mathcal{S}_2$ and $\mathcal{D}$ is determined by the corresponding value-matching and smooth-pasting conditions. That is, applying the ordinary arguments, we obtain

$$F(p,v) = \frac{p}{r-\mu_p} - \frac{(\eta + r - \mu_v)v}{r - \mu_v} + \frac{(r - \mu_v + \eta)v}{(1-\lambda)(r-\mu_v)}\left(\frac{p}{bv}\right)^{\lambda}, \quad \text{for } (p,v) \in \mathcal{S}_2$$

where λ is given in Proposition 1 and the default threshold is

$$b = \frac{\lambda}{\lambda - 1}\frac{r - \mu_p}{r - \mu_v}(r - \mu_v + \eta),$$

which is the same as in (6), whence we obtain $b = b^*$. This means that the default epoch is pushed back to the liquidation time of the pure equity firm in the presence of debt renegotiation. This is so, because debt holders accept the take-it-or-leave-it offer until they do not want to take over the firm. Hence, the boundary $\mathcal{B}_{\mathcal{DS}_2}$ between $\mathcal{S}_2$ and $\mathcal{D}$ is given by the straight line $p/v = b^*$. As soon as the process (P, V) hits this boundary, debt holders stop the renegotiation and liquidate the firm immediately.

Similarly, the third equation in (18) has a closed form solution and the boundary $\mathcal{B}_{S_1S_2}$ between $\mathcal{S}_1$ and $\mathcal{S}_2$ is already known as in (17).[8] Hence, we obtain

$$F(p,v) = \frac{(1-\xi)p}{r-\mu_p} + \frac{(1-\xi^\lambda)(r-\mu_v+\eta)v}{\xi^\lambda(1-\lambda)(r-\mu_v)}\left(\frac{p}{\underline{b}v}\right)^\lambda, \quad \text{for } (p,v) \in \mathcal{S}_1$$

where $\underline{b}$ is given by (14).

The boundary $\mathcal{B}_{CL}$ ($\mathcal{B}_{CD}$, respectively) between $\mathcal{L}$ ($\mathcal{D}$) and $\mathcal{C}$ should be obtained as a part of the free-boundary problem for the first (fifth) and second equations in (18). It remains to determine the boundary $\mathcal{B}_{CS}$ between $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{C}$. This boundary is selected by equity holders so as to minimize the debt value $D(p,v)$, where $D(p,v)$ is equal to $X(p,v)$ in the region $\mathcal{S}$.[9] Moreover, from the fact that $W(p,v) = F(p,v) + D(p,v)$ and the firm value $W(p,v)$ is irrelevant to the existence of debt renegotiation (see Equation (20)), minimization of $D(p,v)$ is equivalent to maximization of $F(p,v)$. Therefore, the boundary $\mathcal{B}_{CS_1}$ ($\mathcal{B}_{CS_2}$, respectively) is obtained as a part of the free-boundary problem for the second and third (fourth) equations in (18).

Next, the debt value $D(p,v)$ satisfies the following PDE:

$$\begin{cases} D(p,v) = \dfrac{c}{r}, & \text{for } (p,v) \in \mathcal{L}, \\ \mathcal{A}D(p,v) + c = 0, & \text{for } (p,v) \in \mathcal{C}, \\ D(p,v) = X(p,v), & \text{for } (p,v) \in \mathcal{S}, \\ D(p,v) = v, & \text{for } (p,v) \in \mathcal{D}. \end{cases} \tag{19}$$

Given the boundaries that have been obtained by solving the free-boundary problem (18), the second equation must satisfy the value-matching condition to the other equations on the boundaries.

Finally, the firm value is defined by $W(p,v) = F(p,v) + D(p,v)$, so that it satisfies the following PDE:

$$\begin{cases} W(p,v) = v, & \text{for } (p,v) \in \mathcal{L}, \\ \mathcal{A}W(p,v) + p - \eta v = 0, & \text{for } (p,v) \in (\mathcal{L} \cup \mathcal{D})^c, \\ W(p,v) = v, & \text{for } (p,v) \in \mathcal{D}. \end{cases} \tag{20}$$

Here, the second equation must satisfy the value-matching condition to the other equations on the boundaries.

As before, we solve the PDEs (18) and (19) numerically by using a finite difference method. For this purpose, it is helpful to derive the value functions in closed form for v, $p \to 0$ and for

[8] Of course, the boundary $\mathcal{B}_{S_1S_2}$ (straight line $p/v = \underline{b}$) is derived by solving the corresponding value-matching and smooth-pasting conditions of the third and fourth equations in (18).

[9] Mella-Barral and Perraudin (1997) derive the boundary $\mathcal{B}_{CS}$ by using no-arbitrage arguments. However, it can be readily verified that the no-arbitrage condition is equivalent to minimizing the debt value.

Table 1: Base Parameters

σ_p	σ_v	μ_p	μ_v	ρ	η	ξ	r	c
30%	15%	4%	2%	0.7	1%	0.7	6%	8%

sufficiently large p and/or v (see Goto et al., 2010). Note that, because strategic debt service is never performed for sufficiently large p and/or v, the value functions there are the same as those without renegotiation.

3.2 Numerical Analyses

In this section, we provide numerical examples to explain how strategic debt service is performed. Figure 1 depicts the calculation results for the base case with parameters listed in Table 1, where the four regions (operating $\mathcal{C}$, debt renegotiation $\mathcal{S}$, default $\mathcal{D}$ and liquidation $\mathcal{L}$) are obtained by solving the free-boundary problem (18). We employ the finite difference method with successive over-relaxations in terms of the transformed variables (x, y) defined by

$$x = \frac{p}{p+1}, \qquad y = \frac{v}{v+1}, \tag{21}$$

which are then converted back to the values of (p, v).

In Figure 1, the default region without renegotiation surrounded by a curve in the bottom-left corner is divided into three parts: default region $\mathcal{D}$ above the line $p/v = b^*$, renegotiation region $\mathcal{S}_1$ with service flow $\xi p - \eta v$ below the line $p/v = \underline{b}$, renegotiation region $\mathcal{S}_2$ with service flow $(r - \mu_v)v$ between $\mathcal{D}$ and $\mathcal{S}_1$. The dashed and chained lines in Figure 1 represent the straight lines $p/v = b^*$ and $p/v = \underline{b}$, respectively. The points $c/r = 1.3333$, $\hat{U} = 1.4693$ and $\hat{L} = 1.2220$ on the v-axis and the point $K = 0.01996$ on the p-axis are also appended there.

In Figure 1, we add typical sample paths of the process (P, V) in order to explain the renegotiation/default/liquidation mechanism in our model. For example, starting from a point (p, v) in the operating region $\mathcal{C}$;

1. The process (P, V) hits the region $\mathcal{S}_1$ before default and liquidation. In this case (sample path (a)), equity holders offer the service flow $\xi p - \eta v$ and debt holders accept it. The firm is operated by equity holders. After that, the service flow may change to $(r - \mu_v)v$ when the process (P, V) hits the line $p/v = \underline{b}$. Moreover, the firm may be liquidated by debt holders if the process (P, V) hits the line $p/v = b^*$. If the process (P, V) returns to the region $\mathcal{C}$, the service flow goes back to the contractual coupon c and the debt renegotiation is reset.

2. The process (P, V) first crosses the line $p/v = \underline{b}$ and then hits the region $\mathcal{S}_2$. In this case (sample path (b)), equity holders offer the service flow $(r - \mu_v)v$ and debt holders accept it. After that, the firm may be liquidated if the process (P, V) hits the line $p/v = b^*$. If the process (P, V) returns to the region $\mathcal{C}$, the service flow goes back to c and the debt renegotiation is reset.

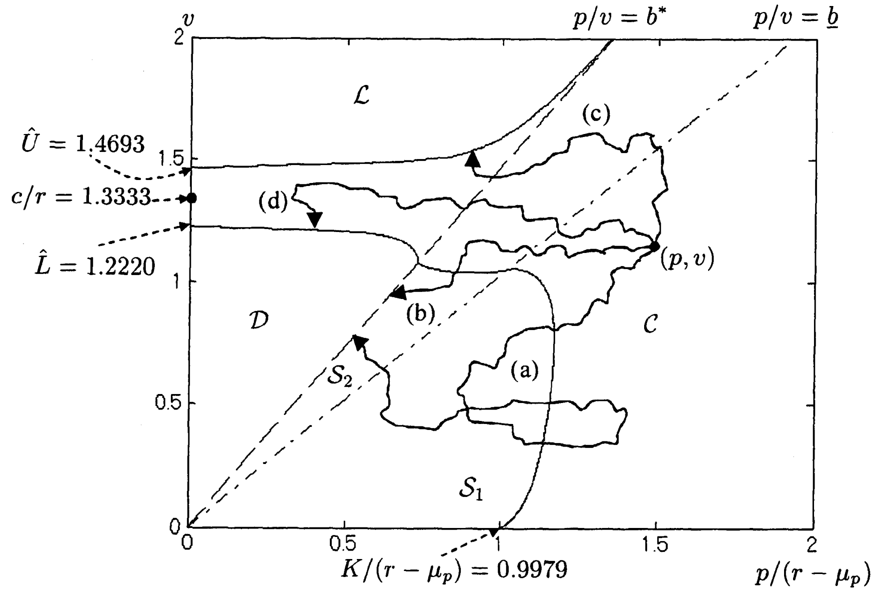

Figure 1: Default, liquidation and renegotiation boundaries for the base case.

3. The process (P, V) first crosses the line $p/v = b^*$ and then hits the liquidation region $\mathcal{L}$ without visiting the default region $\mathcal{D}$. In this case (sample path (c)), equity holders liquidate tangible assets and repay the face value of debt to debt holders. Since liquidation occurs above the line $p/v = b^*$, it is a late liquidation.

4. The process (P, V) first crosses the line $p/v = b^*$ and then hits the default region $\mathcal{D}$ without visiting the liquidation region $\mathcal{L}$. In this case (sample path (d)), equity holders default and debt holders liquidate the firm immediately. Since liquidation occurs above the line $p/v = b^*$, it is a late liquidation.

3.2.1 Significance of tangible assets

First, our model admits a late liquidation, because there is an operating region beyond the line $p/v = b^*$ between default region $\mathcal{D}$ and liquidation region $\mathcal{L}$. Recall that this gap shrinks and diminishes as the volatility σ_v of tangible assets tends to zero.

The gap between default and liquidation regions can also produce the possibility of *liquidity default*. As Hart and Moore (1994, 1998) emphasize, it is important to distinguish liquidity default from strategic default. While strategic default occurs when the firm fails to pay full amount of debt service in debt contract even though it possesses the resource to do so, liquidity default occurs when the firm's cash flows are insufficient to cover the debt service. In our model, default following sample path (d) in Figure 1 corresponds to liquidity default, because default occurs when the value of tangible assets is relatively high and the EBIT is substantially low. In this case, the firm must pay a high maintenance cost for the tangible assets despite of the low earnings, so that equity holders cannot afford to carry out the debt contract, resulting in liquidity

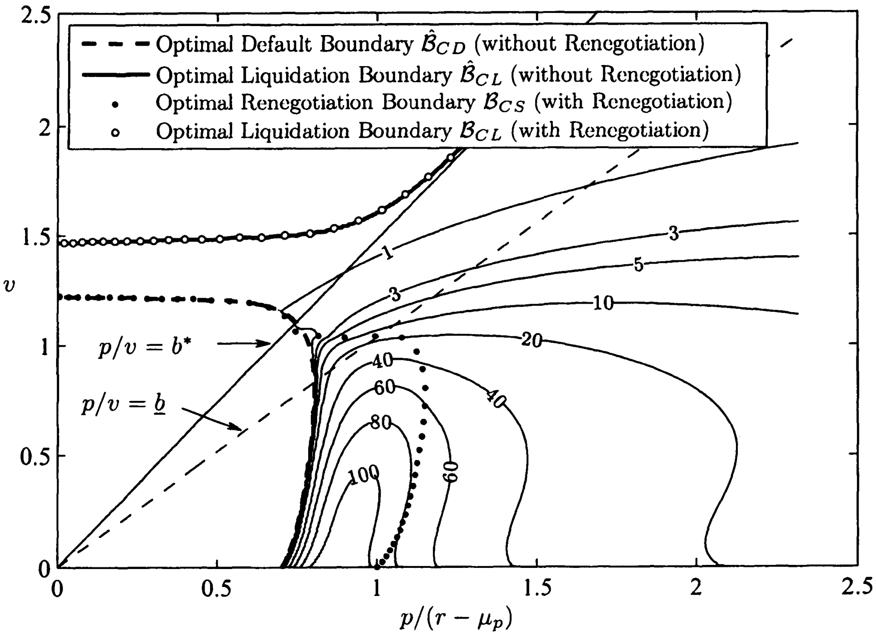

Figure 2: Contribution of strategic debt service to credit spreads.

default. Our model can treat strategic and liquidity defaults within the same framework.

3.2.2 Contribution of strategic debt service to credit spreads

Using numerical examples, we investigate how much strategic debt service contributes to credit spreads. Recently, Davydenko and Strebulaev (2007) showed that (i) the corporate bond prices are affected by the possibility of renegotiation, especially when the costs of liquidation are likely to be high, and (ii) changes in corporate bond price due to renegotiation are high, especially when the credit quality of the issuer is relatively row. In our model, (i) high cost of liquidation means a low value of tangible assets, and (ii) low credit quality means a small value of tangible assets or small value of EBIT. The additional credit spreads shown in Figure 2 support these empirical findings.

In Figure 2, the contour lines show the additional credit spreads (bp), defined by

$$\frac{c}{D(p,v)} - \frac{c}{\hat{D}(p,v)},$$

due to the debt renegotiation. The bold dashed and solid lines plot the optimal default and liquidation boundaries without renegotiation, respectively, while the white and black circles plot the optimal liquidation and renegotiation boundaries with renegotiation, respectively. The parameters are taken from Table 1. It is explicitly observed that the additional credit spreads are high when the value of tangible assets is low and the value of EBIT is small. Note that, in Mella-Barral and Perraudin (1997), additional credit spreads do not depend on the value of EBIT.

Davydenko and Strebulaev (2007) also showed that quantitative contribution of strategic debt service to credit spreads is below transaction costs, assuming that creditors have no bargaining power in their pricing model. This result contradicts the results shown by the theoretical models of debt renegotiation, including Mella-Barral and Perraudin (1997), Fan and Sundaresan (2000), and Anderson and Sundaresan (1996). These theoretical models suggest that, when creditors have little bargaining power, a large part of credit spreads may be due to the possibility of renegotiation of debt service. Hence, based on the theoretical results, Davydenko and Strebulaev (2007) conclude that bondholders are likely to have significant bargaining power. However, our model suggests that, even though equity holders have full bargaining power, the additional credit spreads due to debt renegotiation are significantly low compared to the existing theoretical results, as is shown in Figure 2. This is because our model considers the possibility of liquidity default, which allows no opportunity of renegotiation for equity holders, despite that they have full bargaining power.

4 Conclusion

In this paper, we propose a new pricing model for corporate securities issued by a levered firm with the possibility of debt renegotiation. We take the structural approach that the firm's earnings follow a geometric Brownian motion with stochastic collaterals. As in Leland (1994), equity holders can default the firm for their own benefits, when the earnings become insufficient to go on the firm. In addition, equity holders may want to liquidate the firm by repaying the face value of debt to debt holders to get enough residuals, when the value of collaterals becomes sufficiently high. Unlike the existing theoretical models, the bivariate structure can not only capture realistic credit spreads observed in the market, but also explain many empirical findings reported in the literature.

As future studies, it seems important to consider the optimal capital structure of a firm in the bivariate setting. To this end, we need to develop an efficient numerical method to solve the associated bivariate free-boundary problems, as mentioned in Remark 1. Furthermore, we want to relax Assumption 2 to be more realistic. That is, consider the situation that raiders choose the acquisition timing so as to maximize their return under imperfect competitive market. It seems interesting to investigate how equity holders' strategy changes according to the change of liquidation time.

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